Table 2 CPU timings for major steps in analysis and DSA for pressurized cylinder example

Step	Full sensitivity ID	Full sensitivity Hybrid	Partial sensitivity ID	Partial sensitivity Hybrid
Analysis				
Preliminaries	3.5	4.1	2.5	2.7
Numerical integration	211.2	211.6	152.1	152.9
Zone assembly	0.5	0.3	0.4	0.4
Overall assembly	10.4	17.4	7.7	15.2
Matrix factorization	100.4	99.5	72.3	74.6
Forward reduction & back substitution	5.7	5.5	4.1	4.4
Surface stress recovery	7.3	7.7	5.5	5.6
Reanalysis				
Preliminaries	0.0	0.0	0.0	0.1
Numerical integration	0.0	197.7	0.0	156.6
Assembly	0.0	0.2	0.0	0.2
Iterations	0.0	21.1	0.0	20.1
Surface stress recovery	0.0	5.7	0.0	5.4
Design sensitivity analysis				
Preliminaries	0.1	0.0	0.1	0.0
Numerical integration	281.5	0.6	151.2	0.5
Zone assembly	2.3	0.0	1.9	0.0
Overall assembly	9.9	0.0	7.4	0.0
Forward reduction & back substitution	9.8	1.2	6.8	1.1
Surface stress recovery	7.7	5.7	5.7	5.7
Total	650.3	578.3	417.7	445.5

sensitivity analyses were performed as the originally circular hole was modified into an elliptical shape of various aspect ratios. Thus, the design variable $a = X_L$ in this problem is the (horizontal) semimajor axis of the elliptical hole. The BEA model was made up of 28 three-node quadratic elements and 56 nodes. This example took five simple iterations to converge during the reanalysis process. As seen in Table 1b, the sensitivities of the derived stress components at the sample points distributed around the hole are in good agreement with the predictions computed using a direct boundary element shape sensitivity analysis of a model with this same geometry.

Conclusions

Shape sensitivity of continuum structural models has been treated by a new hybrid method that does not involve matrix derivatives. This approach represents a highly attractive alternative to the implicit differentiation method in which matrix derivatives are used because of the ease with which it can be implemented in general BEA computer programs. The accuracy and computational efficiency associated with this new method seem to be quite competitive with other methods.

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Dynamic Continuum Plate Representations of Large Thin Lattice Structures

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Introduction

TRUSS-TYPE lattice structures have been the dominant form proposed for large space structures (LSS). Conventional finite element analysis of LSS may require a significant amount of storage capacity and computing time to obtain reliable solutions because of high structural flexibility and large size, especially in the dynamic analysis. Thus, alternative methods have been developed for simplified structural modeling of lattice structures. Of these methods, the continuum methods based on energy equivalence[1] have been shown to give satisfactory results when the wavelength of a vibration mode spans many repeating cells of a lattice structure. The author has proposed a new energy equivalence technique for the large platelike lattice structures,[1] where the idea of using conventional finite element matrices was utilized in the calculation of strain and kinetic energies stored in a repeating cell. The objective of the present Note is to develop classical thin plate continuum models of large platelike lattice structures, where the effects of transverse shear deformations and rotary inertia are negligible. For continuum modeling, three basic assumptions for the lattice plate are made: 1) it behaves grossly as a continuum plate, especially in lower mode vibrations; 2) the ratio of in-plane dimensions to thickness is extremely high; and 3) the deflection is very small compared to the thickness. In this Note, the lattice plate is transformed to the continuum plate by following the same procedure as introduced in Ref. 1. For further discussion, the readers are referred to the figures and nomenclatures used in Ref. 1.

Reduced Equivalent Continuum Stiffness and Mass Matrices for a Lattice Plate

We revisit rectangular lattice plates having different types of repeating cells as shown in Ref. 1. The repeating cells are composed of several different types of lattice elements, and all joints will be considered as nodal points on which nodal displacement vectors are defined. The nodal displacement vector at ith node of (x_i, y_i, z_i) is defined by

$$\{\delta_i\} = \{u_i \quad v_i \quad w_i\}^T \tag{1}$$

Three rotational displacements can be readily added to $\{\delta_i\}$ for the nonhinged-type joints. Neglecting midsurface stretch-

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ing u and v, the nodal displacement vector at $(x_i, y_i, 0)$ is defined by

$$\{\bar{\delta}_i\} = \{\bar{w}_i \quad \bar{\theta}_{xi} \quad \bar{\theta}_{yi}\}^T \tag{2}$$

We now introduce the continuum DOFs defined at the center of midsurface (0, 0, 0) as

$$\{\Delta\} = \{w_0 \quad \theta_{x0} \quad \theta_{y0} \mid \kappa_x \quad \kappa_y \quad \kappa_{xy}\}^T = \{\Delta^R \mid \Delta^E\}^T \tag{3}$$

where $\{\Delta^R\}$ is the rigid-body displacements vector and $\{\Delta^E\}$ is the elastic curvatures vector at (0, 0, 0), as defined in Ref. 1.

For small amplitude vibration, approximated relations between $\{\delta_i\}$ and $\{\Delta\}$ and between $\{\bar{\delta}_i\}$ and $\{\Delta\}$ can be derived by use of Taylor's series expansion as follows:

$$\{\delta_i\} = [A_i]\{\Delta\}, \qquad \{\bar{\delta}_i\} = [B_i]\{\Delta\} \tag{4}$$

The transformation matrices $[A_i]$ and $[B_i]$ are given in Table 1.

By using Eqs. (3) and (4), two nodal displacement vectors at two ends of the eth lattice element are combined to construct a displacement vector in the form of

$$\{d_e\} = \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [X_e]\{\Delta^R\} + [Y_e]\{\Delta^E\} \tag{5}$$

By using the finite element stiffness matrix $[k_e]$ and mass matrix $[m_e]$ for a lattice element with respect to global coordinates, the strain and kinetic energies stored in the lattice element can be obtained approximately as follows:

$$V_e = ½\{d_e\}^T[k_e]\{d_e\}, \qquad T_e = ½\{\dot{d}_e\}^T[m_e]\{\dot{d}_e\} \tag{6}$$

Summing all element energies of Eq. (6), total energies stored in a repeating cell are now calculated from

$$\begin{aligned} V &= \sum_e V_e = ½\{\Delta^E\}^T[K_L]\{\Delta^E\} \\ T &= \sum_e T_e = ½\{\dot{\Delta}^R\}^T[M_L]\{\dot{\Delta}^R\} \end{aligned} \tag{7}$$

We assumed that the rigid-body displacements vector $\{\Delta^R\}$ contributes only to the kinetic energy, whereas the elastic curvatures vector $\{\Delta^E\}$ contributes only to the strain energy. The matrices $[K_L]$ and $[M_L]$ of Eq. (7) are the reduced equivalent continuum stiffness amd mass matrices for a lattice plate given as

$$[K_L] = \sum_e [Y_e]^T[K_e][Y_e], \qquad [M_L] = \sum_e [X_e]^T[M_e][X_e] \tag{8}$$

Table 1 Transformation matrices $[A_i]$ and $[B_i]$

$[A_i]$	$[B_i]$
$\begin{bmatrix} 0 & 0 & z_i \\ 0 & -z_i & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & y_i & -x_i & -½x_i^2 & -½y_i^2 & -½x_iy_i \\ 0 & 1 & 0 & 0 & -y_i & -½x_i \\ 0 & 0 & 1 & x_i & 0 & ½y_i \end{bmatrix}$

Table 2 Matrices $[k_5]$ and $[k_6]$ in Eq. (13)

$$[k_5] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & c \\ & 0 & -a & 0 & 0 & a & 0 & 0 & -a & 0 & 0 & a \\ & & b & 0 & a & 0 & c & -a & b & -c & a & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & -c \\ & & & & 0 & -a & 0 & 0 & a & 0 & 0 & -a \\ & & & & & -b & -c & a & 0 & c & -a & -b \\ & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & 0 & -a & 0 & 0 & a \\ & & \text{symmetric} & & & & & & b & 0 & a & 0 \\ & & & & & & & & & 0 & 0 & 0 \\ & & (a = 2.5,\ b = 15,\ c = 30) & & & & & & & & 0 & -a \\ & & & & & & & & & & & -b \end{bmatrix}$$

$$[k_6] = \begin{bmatrix} 0 & 0 & 0 & 0 & -c & 0 & 0 & c & 0 & 0 & 0 & 0 \\ & b & -a & c & 0 & a & -c & b & -a & 0 & 0 & a \\ & & 0 & 0 & a & 0 & 0 & -a & 0 & 0 & a & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 \\ & & & & -b & -a & 0 & 0 & a & c & -b & -a \\ & & & & & 0 & 0 & a & 0 & 0 & -a & 0 \\ & & & & & & 0 & 0 & 0 & 0 & c & 0 \\ & & & & & & & b & -a & -c & 0 & a \\ & & \text{symmetric} & & & & & & 0 & 0 & a & 0 \\ & & & & & & & & & 0 & 0 & 0 \\ & & (a = 2.5,\ b = 15,\ c = 30) & & & & & & & & -b & -a \\ & & & & & & & & & & & 0 \end{bmatrix}$$

Reduced Finite Element Matrices for Homogeneous Anisotropic Continuum Plates

We consider a rectangular four-noded finite element of a homogeneous anisotropic plate. Assuming that the finite element is placed on the midsurface of a lattice repeating cell, we define nodal displacement vector $\{\bar{\delta}_i\}$ at each nodal point. The nodal displacement vectors can be combined to construct a displacement vector $\{q\}$ in the form of

$$\{q\} = \{\bar{\delta}_1 \quad \bar{\delta}_2 \quad \bar{\delta}_3 \quad \bar{\delta}_4\}^T = [Z]\{\Delta^R \mid \Delta^E\}^T \tag{9}$$

Based on the classical thin plate theory, the strain and kinetic energy of a finite element can be obtained from

$$\begin{aligned} V &= \frac{1}{2}\iint \{\Delta^E\}^T[D]\{\Delta^E\}\ \mathrm{d}x\mathrm{d}y \\ T &= \frac{1}{2}\iint \rho\dot{w}^2\ \mathrm{d}x\mathrm{d}y \end{aligned} \tag{10}$$

Where $\rho(x, y)$ is the mass density per unit area and D_{ij} $(i,j = 1, 2, 6)$ (Ref. 2) are the bending rigidities of an anisotropic plate. The displacement function within a finite element can be expressed as the function of $\{q\}$

$$w = [L(x, y)]\{q\} \tag{11}$$

The shape function matrix $[L]$ is the function of x and y and depends on the coordinates of nodal points. Substitution of Eq. (11) into Eq. (10) leads to the following energy expressions:

$$V = ½\{q\}^T[K]\{q\}, \qquad T = ½\{\dot{q}\}^T[M]\{\dot{q}\} \tag{12}$$

where $[K]$ and $[M]$ are the stiffness and mass matrices for the homogenous anisotropic plate given as follows:

$$\begin{aligned} [K] &= \frac{1}{15ab}[R]\{D_{11}[k_1] + D_{22}[k_2] + D_{12}[k_3] + D_{66}[k_4] \\ &\quad + D_{16}[k_5] + D_{26}[k_6]\}[R] \\ [M] &= \frac{\rho ab}{25200}[R][m][R] \end{aligned} \tag{13}$$

Table 3 Equivalent continuum plate properties

Equivalent continuum properties	Type A		Type B		Type C	
	Present	Sun[5]	Present	Noor[4]	Present	Flower[3]
D_{11}, N-m	2.246×10^7	2.246×10^7	2.364×10^7	2.366×10^7	2.151×10^7	2.151×10^7
D_{22}, N-m	2.246×10^7	2.246×10^7	2.364×10^7	2.366×10^7	2.151×10^7	2.151×10^7
D_{12}, N-m	9.506×10^5	9.500×10^5	6.175×10^6	6.179×10^6	0	0
D_{16}, N-m	0	0	0	0	0	0
D_{26}, N-m	0	0	0	0	0	0
D_{66}, N-m	9.506×10^5	9.500×10^5	6.175×10^6	6.179×10^6	0	0
ρ, kg/m^2	0.221	0.221	0.178	0.178	0.263	0.263

The matrices $[R]$, $[k_1]$, $[k_2]$, $[k_3]$, $[k_4]$, and $[m]$ of Eq. (13) are found to be identical to those for the orthotropic plate,[2] and the matrices $[k_5]$ and $[k_6]$ are new derived herein. Based on the sign convention defined in Fig. 1 of Ref. 1, the matrices $[k_5]$ and $[k_6]$ are tabulated in Table 2. Substituting Eq. (9) into Eqs. (12), the strain and kinetic energies are approximated as follows:

$$V = \tfrac{1}{2}\{\Delta^E\}^T[K_c]\{\Delta^E\}, \qquad T = \tfrac{1}{2}\{\dot{\Delta}^R\}^T[M_c]\{\dot{\Delta}^R\} \tag{14}$$

where

$$[K_c] = ab[D]$$

$$[M_c] = \rho ab \begin{bmatrix} 1 & 0 & 0 \\ 0 & b^2/12 & 0 \\ 0 & 0 & a^2/12 \end{bmatrix} \tag{15}$$

In Eqs. (15), $[K_c]$ and $[M_c]$ are the reduced finite element matrices for a homogeneous anisotropic continuum plate.

Equivalent Continuum Structural Properties and Illustrative Examples

The equivalent continuum properties are now obtained by equating Eq. (8) to Eq. (15), i.e., $[K_L] = [K_c]$ and $[M_L] = [M_c]$. For a homogeneous anisotropic continuum plate model, six bending rigidities and uniform mass density are then obtained from

$$D_{ij} = \frac{1}{ab} K_{Lij} \quad (i,j = 1, 2, 6), \qquad \rho \cong \frac{1}{ab} M_{L11} \tag{16}$$

As referenced in Ref. 1, Flower and Schmidt,[3] Noor et al.,[4] and Sun et al.[5] considered the continuum modeling of three different types of lattice plates. To evaluate the continuum method proposed herein, we revisited the same lattice plates considered by them. Table 3 compares the equivalent continuum plate properties for the three lattice plate models. The numerical results imply that the present continuum method can give very reliable equivalent structural properties compared with other methods. Even the assumption of classical plate theory has greatly simplified the analysis; however, one notes that the present Note deviates from the previous paper[1] as a result of the transverse shear and rotary inertia assumption. Thus, if the transverse shear and rotary inertia effects are considered to be important for a lattice plate, the continuum method introduced in Ref. 1 is recommended instead.

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